SOLUTION OF THE PROBLEM OF THE MOTION OF A VORTEX UNDER THE SURFACE OF A FLUID, FOR FROUDE NUMBERS NEAR UNITY

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The problem of determining the motion of a vortex under the surface of a liquid, under the influence of gravity, for Froude numbers near unity (but larger than unity), may have two solutions, as has been remarked by Moiseev [1]. One of these solutions corresponds to a flow which tends to plane-parallel flow when the intensity of the vortex tends towards zero. The existence of this solution was shown by Ter-Krikorov [2]. The existence of a second solution is shown below. It corresponds to a flow whose free surface tends to a solitary wave form when the intensity of the vortex tends toward zero.

In the first two sections, an approximate method of solution of the problem is expounded. This approximate solution serves to clarify some particularities of the flow. At the same time, it is of use in the actual construction of the exact solution.

1. Formulation of the problem. Consider the motion of a vortex of intensity γ, moving with constant speed c such that the nondimensional speed, or Froude number, $F^2 = c^2/gH$ is near unity, but is larger than unity. The vortex moves in a canal of finite depth H under the surface of an ideal fluid which is under the influence of gravity. We will suppose that far in front of and far behind the vortex the fluid is at rest and that the free surface is then parallel to the bottom of the channel. By reversing the motion we shall consider the flow of fluid past the vortex.

Without loss of generality, we shall suppose that both the speed of the vortex and the depth of the channel are equal to unity.

The system of coordinates will be chosen as is shown in Fig. 1. The vortex occupies the point $A(0, a)$. Letting $w = w(z)$ denote the complex

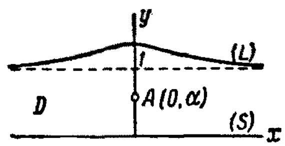

Fig. 1.

potential, the problem reduces to the determination of the function $w(z)$ which is analytic in the domain D, has a logarithmic singularity at the point $z = i\alpha$, and satisfies the following boundary conditions:

$$\frac{1}{2}\left|\frac{dw}{dz}\right|^2 + \nu Y(x) = \text{const} \quad \left(\nu = \frac{1}{F^2}\right), \qquad \psi = 1 \text{ on } (L), \quad \psi = 0 \text{ on } (S) \qquad (1.1)$$

and the asymptotic conditions

$$\lim Y(x) = 1, \qquad \lim\left(\frac{dw}{dz}\right) = 1 \qquad \text{for } |x| \to \infty \qquad (1.2)$$

where $Y = Y(x)$ is the equation of the *a priori* unknown surface of the fluid, and

$$w(z) = \varphi(x, y) + i\psi(x, y) \qquad (z = x + iy)$$

Let us introduce the parametric domain D', the strip of unit width $0 < \eta < 1$. Let us map the domain D in the z plane into the strip D' in the ζ plane by means of the analytic function $\zeta = \xi(x, y) + i\eta(x, y)$ in such a way that the points of infinity correspond, and so that the point $A(0, \alpha)$ in the domain D corresponds to the point $A'(0, \beta)$ in the domain D'. The intensity of the vortex is not affected by the conformal transformation.

Since the curves (L) and (S) are streamlines of the flow in D, it follows that the curves (L') and (S') are streamlines in D'. Thus, in the ζ plane our problem reduces to that of determining the flow past a vortex in a canal of constant width. Hence the complex potential $w(\zeta)$ may be written down explicitly:

$$w(\zeta) = \frac{\gamma}{2\pi i}\ln\frac{\operatorname{sh}[{}^1/{}_2\pi(\zeta - i\beta)]}{\operatorname{sh}{}^1/{}_2\pi(\zeta + i\beta)} + \zeta \qquad (1.3)$$

where

$$\operatorname{Im} w(\zeta) = 0 \quad \text{for } \eta = 0, \qquad \operatorname{Im} w(\zeta) = 1 \quad \text{for } \eta = 1 \qquad (1.4)$$

Inserting $dw(\zeta)/d\zeta$ in (1.1), we obtain

$$\frac{1}{2}f^2(\xi)\left|\frac{d\zeta}{dz}\right|^2 + \nu y(\xi) = \text{const} \qquad \left(f(\xi) = 1 - \frac{\gamma}{2}\frac{\sin\pi\beta}{\operatorname{ch}\pi\xi + \cos\pi\beta}\right) \qquad (1.5)$$

Clearly, in the present problem we have; const = $1/2 + \nu$.

In order that the speed on the free surface never vanish, it is necessary to require that

$$\gamma < 2\,\mathrm{ctg}\,{}^1\!/_2\,\pi\beta \qquad (1.6)$$

The condition (1.2) now becomes

$$\lim y(\xi) = 1, \quad \mathrm{Im}\,\frac{d\zeta}{dz} = 1, \quad \text{for } |\xi| \to \infty \qquad (1.7)$$

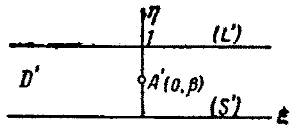

Fig. 2.

Since the axis $y = 0$ corresponds to the axis $\eta = 0$, we have

$$y(\xi,\ 0) = 0 \qquad (1.8)$$

However, the problem in question does not possess a unique solution. Ter-Krikorov [2] constructed an exact solution of this problem, which, as $\gamma \to 0$, tends to the plane parallel flow.

It may be expected that there exist solutions which, as $\gamma \to 0$, tend to a nontrivial solution of the homogeneous problem. The possibility of this nonuniqueness has been pointed out by Moiseev [1]. The unique solution of this problem for $\gamma = 0$ and the given asymptotic conditions at infinity, is a solitary wave.

Consequently, let us pose the problem of determining a solution which, as $\gamma \to 0$, tends to the solution of the homogeneous problem which characterizes a solitary wave. Since a solitary wave exists for Froude numbers near unity, the quantity $1 - \nu$ must be supposed to be small; correspondingly, the quantity γ must also be supposed small, in the general case, of an order not less than that of $1 - \nu$.

2. Approximate solution of the problem. Let us suppose that the equation of the free surface is such that the function $y = y(\xi)$ varies slightly and that its curvature is small. Then, following the plan of the solution in [3], let us replace $d\zeta/dz$ in (1.5) by its approximate expression

$$\left|\frac{d\zeta}{dz}\right|^2 = \frac{1}{y^2(\xi)} + \frac{2}{3}\frac{y''(\xi)}{y(\xi)} - \frac{1}{3}\frac{y'^2(\xi)}{y^2(\xi)} \qquad (2.1)$$

For $u = y(\xi) - 1$ we obtain the equation

$$\begin{aligned} &u''(\xi) - u(\xi)\left\{-3\gamma f_1(\xi) - \frac{9}{2}\gamma^2 f_1^2(\xi) - \cdots + \varepsilon\,[3 + 6\gamma f_1(\xi) + \cdots]\right\} + \\ &+ u^2(\xi)\left\{\frac{9}{2} + 6\gamma f_1(\xi) + 9\gamma^2 f_1^2(\xi) + \cdots - \varepsilon\,[3 + 6\gamma f_1(\xi) + \cdots]\right\} - \\ &- [3\gamma f_1(\xi) + \frac{9}{2}\gamma^2 f_1^2(\xi) + \cdots] - \frac{1}{2}u'^2(\xi) - \frac{1}{2}u(\xi)\,u'^2(\xi) - \cdots = 0 \end{aligned} \qquad (2.2)$$

where

$$\varepsilon = 1 - \nu, \qquad f_1(\xi) = \frac{1}{2}\frac{\sin \pi\beta}{\operatorname{ch} \pi\xi + \cos \pi\beta}$$

The solution of equation (2.2) may be supposed to be of the form $u(\xi) = u_0(\xi) + u_1(\xi)$, where $u_0 = u_0(\xi)$ depends only on ϵ, i.e. is a solution of Equation (2.2) for $\gamma = 0$. The functions $u_0(\xi)$ and $u_1(\xi)$ will be sought in the form

$$u_0(\xi) = \sum_{n=1}^{\infty} \varepsilon^n u_{0n}, \qquad u_1(\xi) = \sum_{n=1}^{\infty} \gamma^n u_{1n} \tag{2.3}$$

where it is required that

$$u_{0n}(\xi) = 0 \quad \text{for } |\xi| \to \infty, \quad u_{1n}(\xi) = 0 \quad \text{for } |\xi| \to \infty \tag{2.4}$$

Substituting from (2.3) into (2.2) and equating to zero the coefficients of ϵ and of γ of u_{01} and of $u_{11}(\xi)$ we obtain the equations

$$u_{01}''(\xi_1) - 3u_{01}(\xi_1) + \frac{9}{2} u_{01}^2(\xi_1) = 0 \tag{2.5}$$

$$u_{11}''(\xi) - 3\varepsilon u_{11}(\xi) + 9u_{01}(\xi_1) u_{01}(\xi) = \Phi(\xi, f_1, u_{01}) \tag{2.6}$$

where

$$\Phi(\xi, f_1, u_{01}) = 3f_1(\xi)[1 + u_{01}(\xi)], \qquad \xi_1 = \sqrt{\varepsilon}\, \xi \tag{2.7}$$

so that Equation (2.6) holds up to order ϵ^2.

The solution of (2.5) satisfying (2.4) has the form

$$u_{01} = \frac{\varepsilon}{\operatorname{ch}^2 \sqrt{3/4\varepsilon}\,(\xi - \xi_0)} \tag{2.8}$$

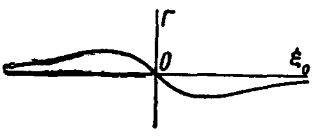

Fig. 3.

In order to integrate (2.6), it is necessary to know two solutions of the homogeneous equation

$$v''(\xi) - 3\varepsilon v + 9u_{01}v = 0 \tag{2.9}$$

These two solutions are

$$v_1 = u_{01}', \qquad v_2 = v_1 \int \frac{d\xi}{v_1^2}$$

The Wronskian of these solutions is

$$\Delta = v_1 v_2' - v_1' v_2 = 1$$

Hence the general solution of (2.6) is given by

$$u_{11}=\frac{1}{\Delta}v_1\int\limits_{-\infty}^{\xi}\Phi(\xi,\ f_1,\ u_{01})\,v_2 d\xi+c_1v_1+\frac{1}{\Delta}v_2\int\limits_{-\infty}^{\xi}\Phi(\xi,\ f_1,\ u_{01})\,v_1 d\xi+c_2v_2 \qquad (2.10)$$

Since $v_2 \to \infty$ as $|\xi| \to \infty$, in order that the function $u_{11}(\xi)$ be bounded in absolute value at infinity we must have $c_2 = 0$ and

$$\Gamma(\xi_0)=\int\limits_{-\infty}^{+\infty}\Phi(\xi,\ f_1,\ u_{01})\,v_1 d\xi=0 \qquad (2.12)$$

Since the function $f_1(\xi)$ is an even function, and the functions $f_1(\xi)$, $u_{01}(\xi)$, $u'_{01}(\xi)$ are monotonic with respect to ξ, the function has the general appearance of the graph in Fig. 3.

From Fig. 3 it follows that $\Gamma(\xi_0) = 0$ only for $\xi_0 = 0$. This leads us to assume that the peak of the wave lies above the vortex.

Since $y(\xi) = 1 + \epsilon u_{01} + \epsilon^2 u_{02} + \ldots + \gamma(u_{11} + \ldots) + \ldots$, as $\gamma \to 0$ the wave tends to a solitary wave, which is a solution of the homogeneous problem, which is characterized by the terms

$$y_1(\xi)=1+\varepsilon u_{01} \qquad \left(u_{01}=\frac{\varepsilon}{\mathrm{ch}^2\sqrt{3/4\,\varepsilon}\,\xi}\right) \qquad (2.13)$$

In what follows, starting from the precise formulation of the problem, we shall prove the following theorem:

when the speed $\nu = e^{-3a^2} < 1$ is near unity (i.e. a is near zero), then for a vortex of small intensity $\gamma = \Gamma / cH$ (where $\gamma < a$) there exists a solitary wave solution.

3. Transformation of the fundamental boundary condition. For the further development of the exact problem and for the simplification of the boundary condition (1.5) we introduce the auxiliary analytic function

$$\omega_1(\zeta)=\theta(\xi,\ \eta)+i\lambda(\xi,\ \eta), \qquad \frac{d\zeta}{dz}=e^{-i\omega_1(\zeta)} \qquad (3.1)$$

Let us substitute (3.1) into (1.5) and differentiate the resulting equation with respect to ξ. Besides, let us observe that

$$\frac{\partial y}{\partial\xi}=\mathrm{Im}\,e^{i\omega_1(\zeta)}=e^{-\tau}\sin\theta, \qquad \frac{\partial\lambda}{\partial\xi}=-\frac{\partial\theta}{\partial\eta}$$

Finally, instead of (1.5) we obtain the following equation:

$$\frac{\partial\theta}{\partial\eta}=\nu\frac{e^{-3\lambda}\sin\theta}{f^2(\xi)}+\frac{f'(\xi)}{f(\xi)} \qquad \left(f(\xi)=1-\frac{\gamma}{2}\frac{\sin\pi\beta}{\mathrm{ch}\,\pi\xi+\cos\pi\beta}\right) \qquad (3.2)$$

Let us set

$$\nu = e^{-3a^2} \quad (\nu \leqslant 1), \qquad \tau = \lambda + a^2, \qquad \omega(\zeta) = \theta(\xi, \eta) + i\tau(\xi, \eta) \quad (3.3)$$

Condition (3.2) is then replaced by the following:

$$\frac{\partial \theta}{\partial \eta} = \frac{e^{-3\tau} \sin \theta}{f^2(\xi)} + \frac{f'(\xi)}{f(\xi)} \tag{3.4}$$

From (3.1) it follows that

$$z = i\alpha + \int_{i\beta}^{\zeta} e^{i\omega_1(t)} dt \quad \text{or} \quad z = i(\alpha - \beta) + \zeta + \int_{i\beta}^{\zeta} [e^{i\omega_1(t)} - 1] dt$$

where a and β are the heights of the submerged vortices in the physical and in the parametric domains D and D' respectively. From the last equation it follows that

$$y = (\alpha - \beta) + \eta + \operatorname{Im} \int_{i\beta}^{\zeta} [e^{i\omega_1(t)} - 1] dt$$

Since $y \to 1$, and $\eta \to 1$ as $|\xi| \to \infty$, we obtain

$$\beta = \alpha + \operatorname{Im} \int_{i\beta}^{-\infty + i} [e^{i\omega_1(t)} - 1] dt \tag{3.5}$$

Condition (1.8) for $\omega(\zeta)$, and also condition (1.7) in view of (3.1), become

$$\lim \omega_1(\zeta) = 0 \quad \text{as} \quad |\xi| \to \infty \tag{3.6}$$

Consequently, from (3.3):

$$\tau \to a^2 \quad \text{as} \quad |\xi| \to \infty \tag{3.7}$$

The exact problem, therefore, may be formulated thus: given a and γ, determine the function $\omega(\zeta) = \theta + i\tau$ which is analytic in the strip $0 < \eta < 1$, is continuous along $\eta = 0$ and $\eta = 1$, and satisfies the boundary conditions (3.4), (3.6) and (3.7), where β is the functional occurring in (3.5).

As was remarked in Section 1, the problem posed possesses a nonunique solution which tends, as $\gamma \to 0$, to a nontrivial solution of the homogeneous problem, characterizing a solitary wave.

We will suppose that $0 < \beta < 1$ and $\nu \leqslant 1$.

The condition $0 < \beta < 1$ means that the vortex does not lie on the free surface, and the condition $\nu \leqslant 1$ means that the vortex moves at a supercritical spead, which is near the critical speed.

4. Green's function. The fundamental boundary condition (3.4) may be rewritten as follows

$$\frac{\partial\theta}{\partial\eta}-\theta=\frac{e^{-3\tau}\sin\theta}{f^3(\xi)}-\theta+\frac{f'(\xi)}{f(\xi)}=F(\theta,\ \tau,\ a,\ \gamma,\ \beta) \tag{4.1}$$

In order to reduce the problem to a nonlinear integral equation, we need the Green's function $G(\zeta,\ \zeta')$ for the strip $0<\eta<1$, satisfying the boundary conditions

$$\frac{\partial H}{\partial\eta}-H=0;\quad \text{for } \eta=1\quad H=0\qquad\qquad \text{for } \eta=0\quad H=\operatorname{Re}G(\zeta,\ \zeta')$$

Such a Green's function $G(\zeta,\ \zeta')$ was first obtained by John [6], and was employed by Friedrichs and Hyers [3].

Lemma 4.1. Suppose that c is a contour consisting of the real axis in the ζ plane, together with a loop around the origin of coordinates in the negative half of the plane.

Then the Green's function $G(\zeta,\ \zeta')$ may be represented as follows

$$G(\zeta,\ \zeta')=\frac{1}{2\pi i}\int\limits_c\frac{\sin\mu(\zeta-i\xi')}{\mu}\,\frac{\mu\operatorname{ch}\mu(1-\eta')-\operatorname{sh}\mu(1-\eta')}{\mu\operatorname{ch}\mu-\operatorname{sh}\mu}\,d\mu \tag{4.3}$$

$$\text{for } 0\leqslant\eta\leqslant\eta'\leqslant1$$

The Green's function may be continued to the remainder of the domain $0<\eta\leqslant1$, $0\leqslant\eta'\leqslant1$ in such a way that conditions (4.2) hold.

Lemma 4.2. The Green's function $G(\zeta,\ \zeta')$ may, for $\eta'=1$, be represented in the form of a sum

$$G(\zeta,\ \zeta')=G_0(\zeta,\ \xi+i)+G_1(\zeta,\ \xi+i) \tag{4.4}$$

where

$$G_0(\zeta,\ \xi+i)=\frac{3}{4}is\left[\frac{1}{5}+(\zeta-\xi')^2\right],\qquad G_1(\zeta,\ \xi+i)=\frac{s}{2\pi}\int\limits_c\frac{e^{-is\mu(\zeta-\xi')}}{\mu\operatorname{ch}\mu-\operatorname{sh}\mu}\,d\mu$$

$$s=\operatorname{sign}(\xi-\xi')=\begin{cases}1 & \text{for } \xi-\xi'>0\\ -1 & \text{for } \xi-\xi'<0\end{cases} \tag{4.6}$$

and, besides

$$G_1(\zeta,\xi'+i)\to0\quad\text{for } |\xi-\xi'|\to\infty \tag{4.7}$$

Lemma 4.3. Suppose that $F(\theta,\ \tau,\ a,\ \gamma,\ \beta)=F(\xi)$ is continuous on $-\infty<\xi<+\infty$ and that the integral

$$\int\limits_{-\infty}^{+\infty}\xi^2F(\xi)\,d\xi<\infty \tag{4.8}$$

exists; then the function

$$\omega(\zeta) = \theta + i\tau = \int_{-\infty}^{+\infty} G(\zeta, \zeta') F(\theta, \tau, a, \gamma, \beta)\, d\xi \qquad (4.9)$$

is an analytic function on the open strip $0 < \eta < 1$, is continuous on the closed strip $0 \leq \eta \leq 1$, and θ satisfies the boundary conditions (3.6) and (4.1), in the sense that

$$\lim(\theta_\eta' - \theta) = F(\xi) \quad \text{as } \eta \to 1,\ \eta < 1; \qquad \theta = 0 \quad \text{for } \eta = 0$$

5. Reduction of the problem to a system of nonlinear integral equations. In order to obtain an equivalent system of nonlinear integral equations, let us introduce the operators

$$GF = \int_{-\infty}^{+\infty} G(\xi, \xi') F(\xi')\, d\xi', \qquad G(\eta) F = \int_{-\infty}^{+\infty} G(\zeta, \xi' + i) F(\xi')\, d\xi' \qquad (5.1)$$

The constructed solution of the nonlinear integral equation

$$\theta + i\tau = GF(\theta, \tau, a, \gamma, \beta) \qquad (5.2)$$

for the functions θ and τ on the boundary $\eta = 1$, furnishes the solution of the problem under consideration. Indeed, according to Lemma 4.3 the solution $\theta + i\tau$ of (5.2) is an analytic function $\omega(\zeta)$, whose boundary values satisfy the first condition (3.6) and condition (4.1).

Let us study the asymptotic character of the functions defined by Formula (5.1), relative to the second condition (3.6).

According to Lemma 4.2 we have

$$G(\zeta, \xi' + i) = G_0(\zeta, \xi + i) + G_1(\zeta, \xi' + i) \qquad (5.3)$$

where

$$G_1(\zeta, \xi' + i) \to 0 \quad \text{as } |\xi - \xi'| \to \infty$$

If $F(\xi)$ is such that the integral (4.8) exists, then, according to Lemma 4.2, $G_1(\eta) F \to 0$ as $|\xi| \to \infty$. But $G_0(\zeta, \xi' + i)$ is the square of a polynomial in ξ and ξ', so that the convergence to zero of $G_0(\eta)F$ as $|\xi| \to \infty$ is not obvious, unless $F(\xi)$ is required to satisfy some additional restrictions. But, since from (4.5)

$$G_0(\zeta, \xi' + i) = \tfrac{3}{4} is (\xi - \xi')^2 + \tfrac{3}{4} \eta s (\xi - \xi') + \tfrac{3}{5} is \left(\tfrac{5}{4}\eta^2 - \tfrac{1}{4}\right) \qquad (5.4)$$

it follows that

$$G_0(\eta) F = -\Omega_1 F \eta + i\Omega_0 F + i\left(\tfrac{5}{4}\eta^2 - \tfrac{1}{4}\right) \Omega_2 F \qquad (5.5)$$

$$\Omega_0 F = \frac{3}{4}\int\limits_{-\infty}^{+\infty} \operatorname{sign}(\xi-\xi')(\xi-\xi')^2 F(\xi')\,d\xi' \tag{5.6}$$

where

$$\Omega_1 F = -\frac{3}{2}\int\limits_{-\infty}^{+\infty} |\xi-\xi'|\,F(\xi')\,d\xi', \qquad \Omega_2 F = \frac{3}{5}\int\limits_{-\infty}^{+\infty} \operatorname{sign}(\xi-\xi')\,F(\xi)\,d\xi'$$

Lemma 5.1. Suppose that $F(\xi)$ is an even function, continuous and decreasing at infinity in such a way that the integral (4.8) exists. If, besides, the integral (5.8) equals zero, then $G_0(\eta)F \to 0$ as $|\xi| \to \infty$ and

$$\Omega_0 F = -3\int\limits_{\xi}^{\infty} d\xi' \int\limits_{\xi'}^{\infty} d\xi'' \int\limits_{\xi''}^{\infty} F(\xi''')\,d\xi'''$$

$$\Omega_1 F = 3\int\limits_{\xi}^{\infty} d\xi' \int\limits_{\xi'}^{\infty} F(\xi'')\,d\xi'', \qquad \Omega_2 F = -\frac{6}{5}\int\limits_{\zeta}^{\infty} F(\xi')\,d\xi' \tag{5.7}$$

The proof is immediate. Since

$$\int\limits_{-\infty}^{+\infty} \xi F(\xi)\,d\xi = 0 \tag{5.8}$$

from (5.6) we readily obtain (5.7).

In view of this we shall suppose that the wave is symmetric, i.e. that r is a even function of ξ and that θ is an odd function of ξ, and then $F(\xi) = F(\theta, r, a, \gamma, \beta)$ will be an odd function of ξ.

Thus, if $\theta(\xi)$ and $r(\xi)$ satisfy (5.8) and (5.2), then by Lemmas 4.3 and 5.1 the function $\omega(\zeta)$ will satisfy the second condition of (3.6); and, consequently, will be a solution of the problem under consideration. The remainder of this paper will be devoted to the determination of the functions θ and r from equation (5.2).

For convenience, we shall separate the operator $G_1 F$ into real and imaginary parts:

$$G_1 F = (T_1 + iT_2)F \tag{5.9}$$

Then the integral equation (5.2), on account of (5.5), (5.7) and (5.9), may be written as follows

$$\theta = [-\Omega_1 + T_1] F(\theta, \tau, a, \gamma, \beta), \qquad \tau = [\Omega_0 - \Omega_1 + T_2] F(\theta, \tau, a, \gamma, \beta) + a^2 \tag{5.10}$$

where β is the functional appearing in (3.5), and

$$\int_{-\infty}^{+\infty} \xi F(\theta, \tau, a, \gamma, \beta)\, d\xi = 0; \qquad \theta \to 0 \text{ as } |\xi| \to \infty, \qquad \tau \to a^2 \text{ as } |\xi| \to \infty \tag{5.11}$$

Suppose that the function $k = k(\xi)$ is odd and such that

$$\int_{-\infty}^{+\infty} \xi k(\xi)\, d\xi = 1$$

Let us introduce the auxiliary function

$$\Phi = \Phi(\theta, \tau, a, \gamma, \beta) = F(\theta, \tau, a, \gamma, \beta) - k(\xi) \int_{-\infty}^{+\infty} \xi F(\theta, \tau, a, \gamma, \beta)\, d\xi \tag{5.12}$$

Then the equation

$$\int_{-\infty}^{+\infty} \xi \Phi(\theta, \tau, a, \gamma, \beta) = 0 \tag{5.13}$$

is identically satisfied.

Instead of the system (5.10) we shall solve the modified system

$$\theta = [-\Omega_1 + T_1] \Phi(\theta, \tau, a, \gamma, \beta), \quad \tau = a^2 + [\Omega_0 - \Omega_2 + T_2] \Phi(\theta, \tau, a, \gamma, \beta)$$

6. Properties of the operators. Let us introduce the classes of continuous functions R_1 and R_2, consisting of odd functions $\theta(\xi)$ and even functions $\tau(\xi)$ respectively, defined on the axis $-\infty < \xi < +\infty$, and such that $|e^{2\xi}\theta(\xi)|$ and $|e^{2\xi}\tau(\xi)|$ are bounded for $\xi > 0$ and consider the norms

$$\|\theta\| = \sup e^{2\xi} |\theta(\xi)|, \qquad \|\tau\| = \sup e^{2\xi} |\tau(\xi)| \qquad (-\infty < |\xi| < +\infty)$$

Let us denote by R the space of pairs $u = \{\theta, \tau\}$ of functions θ and τ, defined on the axis $-\infty < \xi < +\infty$, with the norm

$$\|u\| = [\|\theta\|^2 + \|\tau\|^2]^{1/2}.$$

We now prove a number of lemmas.

Lemma 6.1. If $F(\xi)\in R_1$, then $\Omega_0 F\in R_1$, $\Omega_1 F\in R_1$, $\Omega_2 F\in R_2$, and the following inequalities hold

$$\|\Omega_0 F\| < \tfrac{3}{8}\|F\|, \qquad \|\Omega_1 F\| < \tfrac{3}{4}\|F\| \qquad \|\Omega_2 F\| < \tfrac{3}{5}\|F\| \tag{6.1}$$

Proof: Since

$$\Omega_0 F = -3\int_\xi^\infty d\xi' \int_{\xi'}^\infty d\xi'' \int_{\xi''}^\infty F(\xi''')\,d\xi'''$$

we have

$$|\Omega_0 F| < 3\|F\| \int_\xi^\infty d\xi' \int_{\xi'}^\infty d\xi'' \int_{\xi''}^\infty e^{-2\xi'''}d\xi''' = \frac{3}{8}\|F\| e^{-2\xi}$$

From this, $\|\Omega_0 F\| < 3/8\, F$. The last two inequalities of (6.1) are proved in the same way.

Lemma 6.2. If $F(\xi)\in R_1$, then $T_1F\in R_1$ and $T_2F\in R_2$ and the following inequalities hold

$$\|T_1 F\| < q\|F\|, \qquad \|T_2 F\| < q\|F\| \qquad (q = \text{const}) \tag{6.2}$$

As we shall see below, $q \geqslant 0.9065$.

Proof: Let

$$J = (T_1 + iT_2)\,F = \frac{s}{2\pi}\int_{-\infty}^{+\infty}\left[\int_c \frac{e^{-s\mu i(\xi-\xi')}e^{s\mu}}{\mu\,\mathrm{ch}\,\mu - \mathrm{sh}\,\mu}\,d\mu\right]F(\xi')\,d\xi' \tag{6.3}$$

Using Euler's formula and the oddness of $F(\xi)$ it may be shown that $J = J_1 + J_2$, where

$$J_1 = -\frac{1}{\pi}\int_c \frac{\mathrm{ch}\,(\mu - i\mu\xi)}{\mu\,\mathrm{ch}\,\mu - \mathrm{sh}\,\mu}\,d\mu \int_\xi^\infty e^{-i\mu\xi'}F(\xi')\,d\xi'$$
$$J_2 = \frac{i}{\pi}\int_c \frac{e^{\mu}e^{-i\mu\xi}}{\mu\,\mathrm{ch}\,\mu - \mathrm{sh}\,\mu}\,d\mu \int_0^\xi \sin\mu'\xi' F(\xi')\,d\xi' \tag{6.4}$$

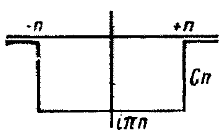

Fig. 4.

Put $\mu = \sigma - i\lambda$, consider the contour C_n in the $\lambda > 0$ half plane, and let us estimate J_1 by deforming C_n around the poles of the integrand

$$\mu_m\,\mathrm{ch}\,\mu_m - \mathrm{sh}\,\mu_m = 0$$

Let us write J_1 as follows:

$$J_1 = -\frac{1}{2\pi}\int_c \frac{e^{\mu(1-i\xi)} + e^{-\mu(1-i\xi)}}{\mu\,\mathrm{ch}\,\mu - \mathrm{sh}\,\mu}\,d\mu \int_\xi^\infty e^{-i\mu\xi'}F(\xi')\,d\xi' \tag{6.5}$$

Estimating the inner integral in (6.5) we obtain

$$\left|\int_{\xi}^{\infty} e^{-i\mu\xi'}F(\xi')\,d\xi'\right| < \|F\|\left|\int_{\xi}^{\infty} e^{-2\xi'}e^{-i(\sigma-i\lambda)\xi'}d\xi'\right| = \frac{1}{2+\lambda}\|F\|e^{-2(1+{}^1/_2\lambda)\xi} \tag{6.6}$$

Let us put

$$K_1(\mu,\xi) = \frac{e^{-i\mu\xi}}{2}\int_{\xi}^{\infty} e^{-i\mu\xi'}F(\xi')\,d\xi', \qquad K_2(\mu,\xi) = \frac{e^{i\mu\xi}}{2}\int_{\xi}^{\infty} e^{-i\mu\xi'}F(\xi')\,d\xi'$$

Then, letting $n \to \infty$, it is easily seen that

$$J_1 = 2\pi i\sum R_{m_1} + 2\pi i\sum R_{m2}$$

where

$$R_{m1} = -\frac{1}{\pi}\,\frac{K_1(\mu_m,\xi)(\operatorname{ch}\mu_m + \operatorname{sh}\mu_m)}{\mu_m \operatorname{sh}\mu_m} = -\frac{1}{\pi}\,\frac{1+\mu_m}{\mu_m{}^2}\,K_1(\mu_m,\xi) \tag{6.7}$$

$$R_{m2} = -\frac{1}{\pi}\,\frac{1-\mu_m}{\mu_m{}^2}\,K_2(\mu_m, \tag{6.8}$$

Since $\pi m < \lambda_m < \pi(m + 1/2)$, from (6.6) it follows that

$$|2\pi R_{m1}| < \|F\|\,e^{-2\xi}\,\frac{1+\lambda_m}{\lambda_m{}^2}\,\frac{1}{2+\lambda_m} < \|F\|\,e^{2\xi}\,\frac{1+\pi m+{}^1/_2\pi}{m^2\pi^2}\,\frac{1}{2+\pi}$$

By addition, using (6.8) for all m, we obtain

$$2\pi\sum_{m=1}^{\infty}|R_{m1}| < 0.1929\|F\|e^{-2\xi}, \qquad 2\pi\sum_{m=1}^{\infty}|R_{m2}| < 0.1142\|F\|e^{-2\xi} \tag{6.9}$$

From (6.9) it follows that

$$\|J_1\| < 0.3071\|F\| \tag{6.10}$$

Now let us estimate J_2. The inner integral of J_2 may be estimated thus:

$$\left|\int_0^{\xi} \text{in}\,\mu\xi'F(\xi')\,d\xi'\right| < \frac{1}{2}\|F\|\left\{\frac{e^{(\lambda-2)\xi}-1}{\lambda-2} + \frac{1-e^{-(\lambda+2)\xi}}{\lambda+2}\right\}$$

Let us put

$$K(\mu_m,\xi) = e^{-i\mu\xi}\int_0^{\xi}\sin\mu\xi'F(\xi')\,d\xi'$$

Then

$$|K(\mu,\xi)| < \frac{1}{2}\, e^{-2\xi}\, ||F|| \left\{ \frac{1-e^{-(\lambda-2)\xi}}{\lambda-2} + \frac{e^{-(\lambda-2)\xi}-e^{-2\lambda\xi}}{\lambda+2} \right\}$$

$$J_2 = 2\pi i \sum R_m, \qquad R_m = \frac{i}{\pi}\, \frac{K(\mu_m,\xi)\, e^{\mu_m}}{\mu_m \operatorname{sh} \mu_m} = \frac{i}{\pi} K(\mu_m,\xi)\, \frac{1+\mu_m}{\mu_m{}^2}$$

From this

$$|J_2| = 2\pi \sum_{m=1}^{\infty} |R_m| < \sum_{m=1}^{\infty} \left\{ e^{2\xi}\, ||F|| \left[\frac{1}{\lambda_m - 2} + \frac{1}{\lambda_m + 2} \right] \frac{1+\lambda_m}{\lambda_m{}^2} \right\}$$

But, since $\pi m < \lambda_m < \pi(m + 1/2)$, we obtain

$$|J_2| < \|F\|\, e^{-2\xi} \sum_{m=1}^{\infty} \frac{1}{m^2}\left(\frac{7.3}{2.15} + \frac{1}{m}\right) < 0.599\,\|F\|\, e^{-2\xi}\, \|J_2\| < 0.5994\,\|F\| \qquad (6.11)$$

From inequalities (6.10) and (6.11) it follows that

$$\|G_1 F\| < q\,\|F\|, \quad q \geqslant 0.9065, \quad \text{or} \quad \|T_1 F\| < q\,\|F\|, \quad \|T_2 F\| < q\,\|F\| \qquad (6.12)$$

and the proof is complete.

Now, (5.14), the fundamental system of integral equations, may be written in the form

$$H(\omega, a, \gamma, \beta) = \{H_1(\omega, a, \gamma, \beta), \quad H_2(\omega, a, \gamma, \beta)\} = 0 \qquad (6.13)$$

where

$$H_1(\omega, a, \gamma, \beta) = \theta - [\Omega_1 + T_1]\, \Phi(\theta, \tau, a, \gamma, \beta)$$

$$H_2(\omega, a, \gamma, \beta) = \tau - a^2 - [\Omega_0 - \Omega_2 + T_2]\, \Phi(\theta, \tau, a, \gamma, \beta)$$

Lemma 6.3. The operator H defined by (6.13) transforms the space R into itself, and possesses, for finite a and γ, and for arbitrary $\omega \in R$, a Frechet differential. The operator H is continuous with respect to ω, a, γ, β, and the differential δH is continuous with respect to ω, a, γ, β, $\delta\omega$, when these variables range over bounded domains or intervals, as the case may be.

Proof: Suppose that $u = \{\theta, \tau\}$ varies in an arbitrary fixed sphere $M \in R$, where $\gamma < 2.\cot\!an(\pi\beta/2)$, and a lies in a finite interval l.

Since the function $F(\xi)$ is given by

$$F(\xi) = \frac{e^{-3\tau}\sin\theta}{f^2(\xi)} - \theta + \frac{f'(\xi)}{f(\xi)} \qquad \left(f(\xi) = 1 - \frac{\gamma}{2}\, \frac{\sin \pi\beta}{\operatorname{ch} \pi\xi + \cos \pi\beta}\right)$$

it is obvious that $F(\xi) \in R$, is an entire function of θ, τ, a, γ, β, which is continuous in ω, a, γ, β under the restrictions placed on a and γ; and that $|F(\xi)| < c$.

Further, from (5.12) it is clear that $\Phi(\theta, \tau, a, \gamma, \beta)$ is also bounded and uniformly continuous with respect to ω, a, γ, β; and that $\Phi \in R_1$, i.e. $\|\Phi\| < c$ for $\omega \in M$, where a, γ, β are finite.

From (6.13) and (6.14), together with Lemmas 6.1 and 6.2, it follows that $H \in R$.

In order to prove the continuity of $H(\omega, a, \gamma, \beta)$, let us notice that

$$H_1(\omega', a', \gamma', \beta') - H_1(\omega, a, \gamma, \beta) = \\ = (\theta' - \theta) + [\Omega_1 - T_1]\{\Phi(\theta', \tau', a', \gamma', \beta') - \Phi(\theta, \tau, a, \gamma, \beta)\} \tag{6.15}$$

Further, using Lemmas 4.1 and 4.2 we see that the norms of the first and second terms of (6.15) may be made small, provided that $\|\theta' - \theta\|$, $\|\tau' - \tau\|$, $|a' - a|$, $|\gamma' - \gamma|$, $|\beta' - \beta|$ are small, and hence H_1 is uniformly continuous in all its variables. This assertion also holds for H_2. Putting

$$\delta\Phi = F_\theta \delta\theta + F_\tau \delta\tau - k(\xi) \int_{-\infty}^{+\infty} \xi \{F_\theta \delta\theta + F_\tau \delta\tau\} d\xi$$

we have that

$$\delta H_1 = \delta\theta - [-\Omega_1 + T_1]\,\delta\Phi(\omega, a, \gamma, \beta;\ \delta\omega)$$
$$\delta H_2 = \delta\tau - [\Omega_0 - \Omega_2 + T_2]\,\delta\Phi(\omega, a, \gamma, \beta;\ \delta\omega)$$

The proof that δH_1 and δH_2 are uniformly continuous in ω, a, γ, β, $\delta\omega$ is similar, and the theorem is proved.

7. Existence theorem. In order to prove the existence of the solution $\omega = \theta + i\tau$ of the equation $H(\omega, a, \gamma, \beta) = 0$, which satisfies

$$\int_{-\infty}^{+\infty} \xi F(\theta, \tau, a, \gamma, \beta)\, d\xi = 0 \tag{7.1}$$

for $\gamma \neq 0$, let us write $\theta = \theta_0 + \delta\theta$, $\tau = \tau_0 + \delta\tau$, where

$$\theta_0 = -\tau_0', \qquad \tau_0 = a^2(1 - 3t_0), \qquad t_0 = \mathrm{sch}^2 \frac{3}{2} a\xi \tag{7.2}$$

is the approximate solution θ, τ in the case of the solitary wave.

Then Equation (6.13) becomes

$$H(\omega_0 + \delta\omega, a, \gamma, \beta) = 0, \quad \text{or} \quad \delta H(\omega_0, a, \gamma, \beta;\ \delta\omega) = \delta\zeta \tag{7.3}$$

where δH is the Frechet differential of the operator H, and $\delta\zeta \in R$.

Consider first the case $\gamma = 0$. In this instance we have the problem of Friedrichs and Hyers. Putting $\xi = a\xi$ and carrying out the indicated operations in (7.3) for $\gamma = 0$, we obtain

$$\delta\theta_1 - g\int_{\xi}^{\infty} d\xi' \int_{\xi'}^{\infty} \{\tau_{01}\delta\theta_1 + \theta_{01}\delta\tau_1 - k(\xi')\Pi(\tau_{01}\delta\theta_1 + \theta_{01}\delta\tau)\}\, d\xi'' = \delta\rho_1$$

$$\delta\tau_1 - g\int_{\xi}^{\infty} d\xi' \int_{\xi'}^{\infty} d\xi'' \int_{\xi''}^{\infty} \{\tau_{01}\delta\theta_1 + \theta_{01}\delta\tau_1 - k(\xi''')\Pi(\tau_{01}\delta\theta_1 + \theta_{01}\delta\tau_1)\}\, d\xi''' = \delta\sigma_1 \tag{7.5}$$

where

$$\delta\theta_1 = \frac{\delta\theta}{a^3}, \qquad \delta t_1 = \delta\tau_1 = \frac{\delta\tau}{a^2}, \qquad \tau_1 = t_1 + 1, \qquad \tau_1 = \frac{\tau}{a^2}, \qquad \theta_1 = \frac{\theta}{a^3}$$

$$\Pi(F) = \int_{-\infty}^{+\infty} \xi F(\xi)\, d\xi$$

Put $y = \delta\tau_1 - \delta\sigma_1$; $-y' = \delta\theta_1 - \delta\rho_1$; then Equation (7.5) becomes

$$y''' - g(\tau_{01}y)' + 9k(\xi)\Pi[(\tau_{01}y)'] + F_1(y) = 0 \tag{7.6}$$

with

$$F_1(y) = 9(\tau_{01}\delta\rho_1 + \theta_{01}\delta\sigma_1 - k(\xi)\Pi(\tau_{01}\delta\rho_1 + \theta_{01}\delta\sigma_1)), \qquad \int_{-\infty}^{+\infty} \xi F_1(y)\, d\xi = 0$$

that is, F_1 is not an arbitrary element of R_1. Integrating (7.6) we obtain

$$y'' - 9\tau_{01}y + ck(\xi) = g(\xi) \qquad c = 9\Pi[(\tau_{01}y)'],\ g(\xi) = \int_{\xi}^{\infty} F_1(\xi)\, d\xi \tag{7.7}$$

Besides, it is readily seen that the function $g(\xi)$ satisfies

$$\int_{-\infty}^{+\infty} g(\xi)\, d\xi = 0$$

In [3] it is shown that Equation (7.7) has solutions in R_2 of the form

$$y = M[g] - cM[K] \tag{7.8}$$

where $M[h]$ is a linear bounded operator. Let us write

$$M[K] = u_1(\xi), \qquad \mu_1(\xi) = \{-u_1(\xi),\ u_1(\xi)\}$$

Lemma 7.1. There exists a linear bounded operator from R to R such that, for each real c and each $\delta J_1 \in R$:

$$\delta\omega_1 = N_1(\delta J_1) - c\mu_1(\xi) \qquad (\omega_1 = \theta_1 + i\tau_1) \tag{7.9}$$

is a solution of the variational Equation (7.5), that is

$$\delta H(\omega_0, 0, 0; \delta\omega_1) = \delta J_1 \tag{7.10}$$

Returning to the variable ξ, Equation (7.9) may be written

$$\delta\omega = N(\delta\zeta) - c\mu \qquad (\delta\omega = 0 \text{ for } a = 0) \tag{7.11}$$

where

$$\mu = \{-a^3 u_1'(a\xi), \quad a^2 u_1(a\xi)\}, \qquad \delta\omega = a^3\delta\theta + ia^2\delta\tau$$

Going back to the equation $H(\omega, a, \gamma, \beta) = 0$, let us write it thus

$$H(\omega_0, 0, 0) - H(\omega, a, \gamma, \beta) = 0 \tag{7.12}$$

Adding $\delta H(\omega_0, 0, 0, z)$ to both sides of (7.12), where $\omega = \omega_0 + z$, we obtain

$$\delta H(\omega_0, 0, 0, z) = T(z, a, \gamma, \beta) + P(a, \gamma, \beta) \tag{7.13}$$

where

$$T(z, a, \gamma, \beta) = \delta H(\omega_0, 0, 0, z) - H(\omega_0 + z, a, \gamma, \beta) + H(\omega_0, a, \gamma, \beta)$$
$$P(a, \gamma, \beta) = H(\omega_0, 0, 0) - H(\omega_0, a, \gamma, \beta) \tag{7.14}$$

Lemma 7.2. Given $\epsilon_1 > 0$, there exist positive numbers l, κ, v, such that

$$\| T(z_1, a, \gamma, \beta) - T(z_2, a, \gamma, \beta) \| < \varepsilon_1 \| z_1 - z_2 \| \tag{7.15}$$

$$\text{for } 0 \leqslant a \leqslant l, \quad \gamma < l, \quad \beta < \varkappa < 1, \quad \| z_1 \| < v, \quad \| z_2 \| < v$$

$$\| P(a, \gamma, \beta) \| < \varepsilon_1 \quad \text{for } 0 \leqslant a \leqslant l, \quad \gamma < l, \quad \beta < \varkappa < 1 \tag{7.16}$$

Proof: We have

$$\begin{aligned} T(z_2, a, \gamma, \beta) - T(z_1, a, \gamma, \beta) &= \delta H(\omega_0, 0, 0;\ z_2 - z_1) - H_0(\omega_0 + z_1, a, \gamma, \beta) + \\ + H(\omega_0 + z_2, a, \gamma, \beta) &= \delta H(\omega_0, 0, 0; z_2 - z) - \delta H(\omega_0 + z_1, a, \gamma, \beta, z_2 - z_1) + \\ &+ \int_0^1 \{\delta H(\omega_0 + z_1, a, \gamma, \beta; z_2 - z_1) - \delta H(\omega_0 + z_1 + s(z_2 - z_1), a, \gamma, \beta; z_2 - z_1)\}\, ds \end{aligned}$$

The Lipschitz condition (7.15) is a consequence of the fact that, in view of Lemma 6.3, the function $\delta H(\omega_0, a, \beta, \gamma; \delta\omega)$ is uniformly continuous in the variables $\omega, a, \beta, \gamma, \delta\omega$, and is continuous in the variables ω, a, β, γ. The assertion (7.16) also follows from Lemma 6.3.

Applying Lemma 7.1 to (7.13), we see that it is equivalent to the equation

$$z = NP(a, \gamma, \beta) + NT(z, a, \gamma, \beta) - c\mu \tag{7.17}$$

For $a = \gamma = 0$, $c = 0$ we have the solution $z = 0$. Since, by Lemma 7.1, N is a bounded linear functional, it follows that the right hand side of (7.17) satisfies a Lipschitz condition with respect to z with a Lipschitz constant which is small for small values of a and γ. Thus, when a, γ, $|c|$ are small, Equation (7.17) may be solved by the usual iteration method, and its solution is

$$\omega(\xi, a, \gamma, \beta, c) = \omega_0 + z(\xi, a, \gamma, \beta, c \tag{7.18}$$

We note that the same can be said in relation to z_c', since z is continuous with respect to c and has a continuous derivative with respect to c, as is easily seen.

It remains to show that c may be chosen, in its dependence on a and γ, such that

$$\Pi(a, \gamma, c) = \int_{-\infty}^{+\infty} \xi F(\theta, \tau, a, \gamma, \beta)\, d\xi = 0$$

However, since for $\gamma = 0$ we have the problem of Friedrichs and Hyers, the function $F(\xi)$ may be represented thus

$$F(\xi) = a^5 F_0(\omega, a, c) + \gamma F_1(\omega, a, \gamma, \beta)$$

Hence

$$\Pi(a, \gamma, c) = \Pi_0(a, c) + \gamma \Pi_1(a, \gamma, c) \tag{7.19}$$

where

$$\Pi_0(a, c) = a^5 \int_{-\infty}^{+\infty} \xi F_0(\omega, a, c)\, d\xi, \qquad \Pi_1(a, \gamma, c) = \int_{-\infty}^{+\infty} \xi F_1(\omega, a, \gamma, \beta)\, d\xi \tag{7.20}$$

Since $\omega_c' = z_c'$ exists and is continuous, (7.19) may be differentiated with respect to c. Consequently

$$\Pi_c' = \Pi_{0c}' + \gamma' \Pi_{1c}'$$

It may be shown that

$$\Pi_{0c}' = a^3 \left(\tfrac{4}{3} + \varepsilon_2\right) \qquad (\varepsilon_2 \to 0 \text{ for } a \to 0).$$

Besides

$$\Pi_{1c}' = \int_{-\infty}^{+\infty} \xi \{F_{1\theta}'\theta_c' + F_{1c}'\tau_c' + \beta_c' [F_{1\beta}' + \theta_\beta' F_{1\theta}' + \tau_\beta' F_{1\tau}]\} d\xi$$

Further, (7.11) implies that

$$\tau_c' = a^2\varphi_1 (\xi, a, \gamma, \beta), \ \theta_c' = a^3\varphi_2 (\xi, a, \gamma, \beta), \ \beta_c' = a^2 v$$

where ϕ_1 and ϕ_2 are bounded functions, and v is a finite number. Thus

$$\Pi_{1c}' = a^2 [A (a, \gamma, \beta) + \varepsilon_3] \qquad (\varepsilon_3 \to 0, \ \gamma \to 0 \text{ for } a \to 0) \qquad (7.20)$$

Therefore

$$\Pi_c' = a^3 (\tfrac{4}{3} + \varepsilon_2) + a^2\gamma [A (a, \gamma, \beta) + \varepsilon_3] \qquad (7.21)$$

Equation (7.21) implies that $\Pi_c'(0, 0, 0,) = 0$ and that $\Pi_c' > 0$ in the neighborhood of $a = 0$, $\gamma = 0$, $c = 0$, provided that $|\gamma| < a$. Thus, under these conditions on γ, the bifurcation equation (7.19) may be uniquely solved for c. In view of this

$$\omega (\xi, a, \gamma, \beta, c) = \omega_0 (\xi, a, c) + z (\xi, a, \gamma, \beta, c) \qquad (7.22)$$

gives a solution of the problem satisfying the condition (7.1).

From the nature of the solution of Equation (7.12) it follows that for small, but fixed, values of a the solution (7.22) tends, as $\gamma \to 0$, to a solitary wave solution.

This concludes the proof of the theorem formulated at the end of Section 2, and our problem has been solved.

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